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Lanczos potential for the Minkowski space

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Abstract : We obtain a Lanczos potential for the space-time of the special relativity with the aid of the Maxwell fields

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The existence of the Lanczos potential K_{ij} to generate the Weyl tensor C_{pqjh} is well established [1–7]. The corresponding expression that relates the two tensors is known to be

$$C_{pqjh} = K_{pqj;h} - K_{pqh;j} + K_{jhp;q} - K_{jhq;p} + g_{ph}K_{jq} - g_{pj}K_{qh} + g_{qj}K_{ph} - g_{qh}K_{pj}, \tag{1}$$

where the tensor K_{ab} is given by

$$K_{ab} \equiv K^{c}_{ab;c} = K_{ba}, \tag{2}$$

g_{ab} is the metric tensor of the Riemannian 4-space and ; denotes covariant derivative.

The Lanczos potential K_{abc} has the algebraic symmetries

$$K_{pqj} = -K_{qpj}, \quad K^{j}_{pj} = 0, \qquad K_{abc} + K_{bca} + K_{cab} = 0, \tag{3}$$

and the differential property (also known as the Lanczos gauge)

$$K^{c}_{ab;c} = 0. \tag{4}$$

It is a difficult task to construct the spintensor K_{ij} of a given metric; however, in [6,8–14], the Newman-Penrose formalism [9,13–16] has been employed to solve eq. (1) for several interesting space-times of gravitation theory.

We devote Section 2 to construct K_{pqj} for the space-time of special relativity where the possible current physical fields do not produce any curvature into the space-time. So, we have $C_{pqjh} = 0$ with $(g_{ab})_{4\times 4} = Diag(1,1,1,-1)$, $(x^1, x^2, x^3, x^4) = (x, y, z, t)$ and the covariant derivative reduces to the ordinary derivative ,. We may mention that Takeno [2] employed a complicated method to construct Lanczos spintensors for this case. Our aim in this communication has been to use some ideas of electrodynamics to find a potential K_{ij} by a simpler and physically more appealing technique.

Lanczos devoted many years, looking for a geometric theory of the electromagnetic field. Once he found the spintensor K_{abc} for the Weyl tensor, he proposed [1] to construct it with the aid of a "*Faraday tensor*" $F_{ab} = -F_{ba}$ through the scheme

$$K_{abc} = F_{ab,c} \tag{5}$$

with F_{ab} satisfying the Maxwell equations in vacuum [17–20] :

$$F^{b}_{a,b} = 0, \quad F_{ab,c} + F_{bc,a} + F_{ca,b} = 0, \tag{6}$$

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so that eq. (3) are obvious; moreover, eqs. (2,5,6) imply that $K_{ab} = 0$. It is simple to verify that when eq. (5) is substituted into eq. (1), we obtain $C_{pqab} = 0$, as expected. Finally, (4,5) lead to the annulation of the D'Alembertian .

$$\Box F_{ab} \equiv F_{ab,c}^{\;\;\;c} = 0 . \tag{7}$$

In summary, with (5) we can construct a spintensor K_{abc} for the Minkowski space provided we are able to find an antisymmetric tensor F_{ab} with the properties implied by eqs. (6,7). Fortunately, we have the interesting result [17,21] :

> *"It is possible to write every solution of (6) in terms of two* (8)
>
> *real wave functions and a constant antisymmetric tensor".* (9)

from which the general solution of (6) can be written down in the form :

$$F_{ab} = \left(H_a{}^c U_{,c} + {}^*H_a{}^c V_{,c}\right)_{,b} - \left(H_b{}^c U_{,c} + {}^*H_b{}^c V_{,c}\right)_{,a} , \tag{10}$$

where U and V satisfy the wave equation

$$\Box U = \Box V = 0, \tag{11}$$

H_{ab} is antisymmetric and constant

$$H_{pq} = -H_{qp}, \quad H_{pq,c} = 0, \tag{12}$$

and $^*H_{ab}$ is the dual of H_{ab} [17–20].

Note that (7) is obtained directly from eqs. (9–11). That is, with the scheme (5), a spintensor of Lanczos for flat R_4 has been generated. F_{ab} is given by (9) with the properties (10,11).

In (9), the tensor (H_{ij}) is a constant antisymmetric 4×4 matrix with respect to the coordinates (x,y,z,t), so, any matrix of this type serves to construct the tensor H_{pq}. Therefore, the problem simplifies into one where one looks for any well-behaved solution of (10).

The inverse of the retarded distance of Lienard-Wiechert [9,17–19,22–25] satisfies the wave equation, nevertheless, this function is singular on the trajectory of the charge, so that we will not employ it, as we want a spintensor without divergences in each event of the space-time.

We know that the function

$$R(x^j) = \frac{1}{(x^r - a^r)(x_r - a_r)} \tag{13}$$

where a^r is any constant vector, satisfies $\Box R = 0$; but it has singularities on the null cone with vertex in a^r. Lanczos [26] (see also [17,27]) showed how to eliminate these divergences; in fact, one just notes that the function :

$$R(x^j - ib^j) \equiv U + iV, \quad i = \sqrt{-1}, \tag{14}$$

where b^r is any constant timelike vector $(b^r b_r < 0)$, also fulfills $\Box R = 0$. Therefore, U and V are wave functions which turn out to be free of singularity for every point in the Minkowski space :

$$U(x^j) = \frac{A}{A^2 + B^2}, \quad V(x^j) = \frac{B}{A^2 + B^2} \tag{15}$$

with the A and B functions given by :

$$A = (x^r - a^r)(x_r - a_r) - b^r b_r, \quad B = 2b^r (x_r - a_r). \tag{16}$$

We do not lose generality if we take $a^r = 0$ and $(b^r) = (0,0,0,b)$ with b a constant $\neq 0$; in that case, eq. (15) would become :

$$A = x^2 + y^2 + z^2 - t^2 + b^2, \quad B = -2bt \tag{17}$$

so that (14) is well-behaved and satisfies eq. (10).

When employing (9,14) in (5), we obtain a spintensor with no singularity in the whole flat space-time.

It is interesting to note that Synge [17], has employed eqs (9,14–16) to develope electromagnetic models of both material particles and photons. In fact, this is the reason for which we believe that spintensor (5) may physically be more relevant than the spintensors reported by Takeno [2].

We have developed the idea that the electromagnetic field allows one to construct objects (spintensors in this case) associated with the geometric structure of the space-time. Another possibility for this idea is

$$K_{abc} = F_{ab}\Phi_c , \tag{18}$$

where F_{ij} is a Faraday tensor and Φ_j is a 4-vector with certain specific properties. It has been employed earlier [18,19] to find the superpotential for the radiative part $\underset{R}{T}_{ab}$ [9,19,22–24,28,29] of the Liénard-Wiechert field.

Furthermore, the spintensor concept has led to important improvements in several cases, for instance, it has been used to assign physical meaning to the Weert potential [30,31], corresponding to the bounded part $\underset{B}{T}_{ab}$ [18,19,24] of the Lienard-Wiechert field, so that one may observe [32,33] .

> *"$\underset{B}{K}_{abc}$ is the density of the intrinsic angular momentum of the electromagnetic field radiated by the charge"* (19)

A result that allows us to suppose that the K_{ijr} built up *via* (5) is associated in some sense, with the angular momentum [17] of the tensor field F_{ab}.

We are investigating this assumption at the present time.

It is worthwhile to mention that nobody (and for any metric) has been able to give a physical meaning to the Lanczos spintensor in general relativity theory; this is an interesting open problem [34–37]. We would like to obtain a result like the one presented in (18) for the Kerr metric [39,40]; this is because we have already obtained [10,38] a special $K_{\mu\nu}$ for this metric and presently, we are studying if such spintensor is related with the black hole rotation.

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